no mark separating the fractional from the integral parts of a number. Hence a number like 44(26)(40) might be interpreted in different ways; among the possible meanings are $44 \times 60^2 + 26 \times 60 + 40$, $44 \times 60 + 26 + 40 \times 60^{-1}$, and $44 + 26 \times 60^{-1} + 40 \times 60^{-2}$. Which interpretation is the correct one can be judged only by the context, if at all.

The exact meaning of the first two lines in the first column is uncertain. In this column 60 is divided by each of the integers written on the left. The respective quotients are placed on the right.

In the fifth column the multiplicand is 44(26)(40) or 44 4/9. The last two lines seem to mean "$60^2 \div 44(26)(40) = 81$, $60^2 \div 81 = 44(26)(40)$."

It is a source of gratification to find that scholars of several thousand years ago were fully as capable of committing errors in computation, as are arithmeticians of the present time.

The Babylonian use of sexagesimal fractions is shown also in a clay tablet described by A. Ungnad[1] (*Orient. Lit. Zeitung,* 19 Jahrg., 1916, p. 363-368). In it the diagonal of a rectangle whose sides are 40 and 10 is computed by the approximation $40 + 2 \times 40 \times 10^2 \div 60^2$, yielding 42(13)(20), and also by the approximation $40 + 10^2 \div \{2 \times 40\}$, yielding 41(15). Translated into the decimal scale, the first answer is 42. 22 +, the second is 41.25, the true value being 41.23 +.

A BUDGET OF EXERCISES ON DETERMINANTS.

By THOMAS MUIR, Rondebosch, South Africa.

A collection of fresh exercises[2] on a mathematical subject, even if the plan and execution be far from perfect, can be of greater service to the student than any so-called "paper," covering the same amount of page space. It is in this belief that I have brought together the following budget of thirty. In a kind of way they range over the whole subject of determinants: at any rate they do not confine themselves to any special branch of it. They are of all degrees of difficulty, starting with commonplace instances of mere "evaluation." They naturally also differ in suggestiveness; one or two of them might in eager hands lead to the formulating of allied results, and thereby even to the evolution of material for a "paper." None of them, so far as I can at present recall, has been printed before, and certainly, the number of them that may so turn out must be comparatively trifling.

1. If $\begin{vmatrix} a_1 - x & a_2 - x & a_3 - x \\ b_1 - x & b_2 - x & b_3 - x \\ c_1 - x & c_2 - x & c_3 - x \end{vmatrix} = 0$, then $x = -\ |a_1b_2c_3| \div \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \end{vmatrix}$.

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[1] Our information is drawn from *Mitteilungen zur Geschichte der Medizin und der Naturwissenschaften,* Leipzig, vol. 17, 1918, p. 203.

[2] For any technical terms, in the following, which the student may have doubt about he is referred to the text-books of Weld and Hanus.

2.
$$\begin{vmatrix} 1 & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ 1 & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \\ 1 & \sin(\alpha+\beta) & \cos(\alpha-\beta) \end{vmatrix} = \begin{vmatrix} \cos 2\alpha & \cos\alpha & \sin\alpha \\ \cos 2\beta & \cos\beta & \sin\beta \\ \cos 2\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

3.
$$\begin{vmatrix} \cos(\beta+\gamma) & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ \cos(\gamma+\alpha) & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \sin(\alpha+\beta) & \cos(\alpha-\beta) \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} \sin 2\alpha & \cos 2\alpha & 1 \\ \sin 2\beta & \cos 2\beta & 1 \\ \sin 2\gamma & \cos 2\gamma & 1 \end{vmatrix}$$

4. Any axisymmetric determinant is expressible as the product of its principal term and an axisymmetric determinant with a diagonal of units.

5. In an n-line zero-axial determinant the number of positive terms is $n - 1$ greater or less than the number of negative terms according as n is odd or even.

6. The reversible continuant $K(x^a x^b x^c x^c x^b x^a x)$ of odd order is resolvable into two continuants.

7. Express the square of the persymmetric determinant

$$\begin{vmatrix} a & b & c & d \\ b & c & d & am \\ c & d & am & bm \\ d & am & bm & cm \end{vmatrix}$$

as a persymmetric.

8.
$$|(a_r + s - 1)^m|_{m+1} = 1^1 2^2 \cdots m^m \zeta^{1/2}(a_1, a_2, \cdots, a_{m+1}).$$

What is written here between the two vertical lines is the (r, s)th element of the determinant, and the suffix outside denotes the order: the first row is thus

$$a_1^m, \qquad (a_1 + 1)^m, \qquad \cdots, \qquad (a_1 + m)^m.$$

When instead of the suffix $m + 1$ we put n, we are faced with a more complicated problem. $\zeta^{1/2}(a_1, a_2, \cdots, a_{m+1})$ denotes the difference product of the a's.

9. If $A = (b + c)(b + d)(c + d)$, $B = (a + c)(a + d)(c + d)$, $\cdots$, then

$$\zeta^{1/2}(A, B, C, D) = (a + b + c + d)^6 \cdot \zeta^{1/2}(a^2, b^2, c^2, d^2).$$

10. If Δ stand for $|a_1 b_2 c_3 d_4|$, then

$$\begin{vmatrix} A_1 - \dfrac{\Delta}{a_1} & A_2 & A_3 & A_4 \\ B_1 & B_2 - \dfrac{\Delta}{b_2} & B_3 & B_4 \\ C_1 & C_2 & C_3 - \dfrac{\Delta}{c_3} & C_4 \\ D_1 & D_2 & D_3 & D_4 - \dfrac{\Delta}{d_4} \end{vmatrix} = \frac{\Delta^3}{a_1 b_2 c_3 d_4} \begin{vmatrix} 0 & a_2 & a_3 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix}$$

11.
$$\begin{vmatrix} abcd & abd & acd & bcd \\ abc & abcd & acd & bcd \\ abc & abd & abcd & bcd \\ abc & abd & acd & abcd \end{vmatrix} = a^3b^3c^3d^3(abcd - \Sigma ab + 2\Sigma a - 3)$$

and the cofactor of $a^3b^3c^3d^3$ on the right is expressible as a continuant and also as a recurrent. What does this cofactor become when the number of variables and the order of the determinant are each made n?

12. In a 3-by-5 array whose row-sums vanish there are three independent linear relations between the determinants of the array.

13. $$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & x_2 & a_3 & a_4 & a_5 \\ -a_1 & -a_2 & x_3 & a_4 & a_5 \\ -a_1 & -a_2 & -a_3 & x_4 & a_5 \\ -a_1 & -a_2 & -a_3 & -a_4 & x_5 \end{vmatrix} = \begin{vmatrix} x_1 & -1 & 0 & 0 & 0 \\ a_1a_2 & x_2 & -1 & 0 & 0 \\ a_1x_2a_3 & a_2a_3 & x_3 & -1 & 0 \\ a_1x_2x_3a_4 & a_2x_3a_4 & a_3a_4 & x_4 & -1 \\ a_1x_2x_3x_4a_5 & a_2x_3x_4a_5 & a_3x_4a_5 & a_4a_5 & x_5 \end{vmatrix}$$
$$= x_1x_2x_3x_4x_5 + \Sigma x_1x_2x_3a_4a_5 + \Sigma x_1a_2a_3a_4a_5.$$

14. Simplify the determinant whose first three columns are taken from those of any 4-by-5 array and whose other column is a column taken from the square of the array.

15. Show that the cofactor of c in

$$\begin{vmatrix} 0 & a & b & c \\ l & 1 & x & yz - x^2 \\ m & 1 & y & zx - y^2 \\ n & 1 & z & xy - z^2 \end{vmatrix}$$

is a factor of the determinant, and find the other factor.

16. Any positive unit orthogonant $|\alpha_1\beta_2\gamma_3|$ is axisymmetric if $1+\alpha_1+\beta_2+\gamma_3$ vanishes or if $\alpha_1 = \beta_2 = \gamma_3 = 1$.

17. If Δ stand for $ad - bc$, then

$$\frac{\partial(a\Delta,\, b\Delta,\, c\Delta,\, d\Delta)}{\partial(a,\, b,\, c,\, d)} = 3\Delta^4.$$

18. The bordered axisymmetric determinant

$$\begin{vmatrix} 0 & x & y & z \\ u & -d+e+f & f & e \\ v & f & d-e+f & d \\ w & e & d & d+e-f \end{vmatrix} = \begin{vmatrix} x & y & z \\ & d-e & d-f \\ & & e-f \end{vmatrix} \cdot \begin{vmatrix} u & v & w \\ & d-e & d-f \\ & & e-f \end{vmatrix}$$

and therefore equals the bordered skew

$$\begin{vmatrix} 0 & x & y & z \\ -u & 0 & d-e & d-f \\ -v & e-d & 0 & e-f \\ -w & f-d & f-e & 0 \end{vmatrix}.$$

19. Eliminate x, y, z, w from the equations got by putting $r = 1, 2, 3, 4$ in

$$a_rxy + b_rxz + c_rxw + d_ryz + e_ryw + f_rzw = 0.$$

20. If

$$\begin{aligned} ax + by + cz &= 0, \\ by + du + ev &= 0, \\ cz + ev + fw &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} mx + ny + pz &= 0, \\ ny + qu + rv &= 0, \\ pz + rv + sw &= 0, \end{aligned}$$

then

$$(mc - ap)(bq - dn)(es - rf) = (dr - qe)(pf - cs)(an - bm).$$

21. $$\begin{Vmatrix} |a_2b_3| & |a_1b_3| & |a_1b_2| \\ |c_2x_3| & |c_1x_3| & |c_1x_2| \\ |y_2c_3| & |y_1c_3| & |y_1c_2| \end{Vmatrix} = \begin{Vmatrix} |b_2c_3| & |b_1c_3| & |b_1c_2| \\ |a_2c_3| & |a_1c_3| & |a_1c_2| \\ |x_2y_3| & |x_1y_3| & |x_1y_2| \end{Vmatrix} = |a_1b_2c_3|\,|c_1x_2y_3|.$$

22. If P be the product of n linear homogeneous functions of n variables, then (H standing for Hessian)

$$\frac{H(P)}{H(\log P)} = -\,(n-1)P^n.$$

23. If the determinant whose columns are the lth, mth, nth, rth columns of any 4-by-8 array be denoted by $|lmnr|$, then

$$\begin{Vmatrix} |1567| & |1568| & |1578| & |1678| \\ |2567| & |2568| & |2578| & |2678| \\ |3567| & |3568| & |3578| & |3678| \\ |4567| & |4568| & |4578| & |4678| \end{Vmatrix} = |1234| \cdot |5678|^3.$$

24. $$\begin{vmatrix} ax & bx + \beta & cx + \gamma \\ bx - \beta & ex & dx + \delta \\ cx - \gamma & dx - \delta & fx \end{vmatrix} = \begin{vmatrix} ax & a\beta & M \\ -a\beta & aPx & N \\ -M & -N & PQx \end{vmatrix} \div a^2P^2,$$

where

$$P = \begin{vmatrix} a & b \\ b & e \end{vmatrix}, \qquad Q = \begin{vmatrix} a & b & c \\ b & e & d \\ c & d & f \end{vmatrix}, \qquad M = \begin{vmatrix} a & b & c \\ b & e & d \\ 0 & \beta & \gamma \end{vmatrix},$$

$$N = \begin{vmatrix} 0 & a & b & c \\ 0 & b & e & d \\ a & 0 & \beta & \gamma \\ b & -\beta & 0 & \delta \end{vmatrix}.$$

25. $$\begin{vmatrix} 0 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 & a_1b_4 + a_4b_1 & \cdots \\ a_1b_2 + a_2b_1 & 0 & a_2b_3 + a_3b_2 & a_2b_4 + a_4b_2 & \cdots \\ a_1b_3 + a_3b_1 & a_2b_3 + a_3b_2 & 0 & a_3b_4 + a_4b_3 & \cdots \\ a_1b_4 + a_4b_1 & a_2b_4 + a_4b_2 & a_3b_4 + a_4b_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n$$

$$= \left\{ -n + 1 - \sum \frac{|a_1b_2|^2}{(-2a_1b_1)(-2a_2b_2)} \right\} \prod(-2a_1b_1).$$

26. The sum of a quadric and its discriminant is expressible as a Pfaffian.

27. The determinant of the sum of an axisymmetric matrix and a zero-axial skew matrix is expressible as a Pfaffian.

28. If D_1, D_2 be n-line determinants, the one axisymmetric and the other zero-axial skew, and Δ_1, Δ_2, $\cdots$ be the series of determinants each of which is identical in m columns with D_2 and in the remaining columns with D_1, then when m is odd

$$\sum\Delta = 0.$$

29. If from the n-line determinants $D_1, D_2, \cdots, D_p$, there be formed a new n-line determinant Δ consisting of α_1 rows from D_1, α_2 rows from D_2, and so on, the rows in each case occupying the same places in Δ as in their own determinant, then

$$\sum\Delta = \sum\Delta',$$

where Δ' is the determinant formed like Δ save that columns are used instead of rows.

30. If A and O be n-line determinants, A axisymmetric and O orthogonant, the sum of the r-line coaxial minors of OAO' is equal to the corresponding sum in A.

Δ' is used to denote the conjugate of Δ, *i.e.*, the determinant got by changing the rows of Δ in order into columns.

February 15, 1921.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

18. Sylvester as a Poet.

The natural relation of the mathematical to the poetical mind has been observed so often as to make it hardly worth while to comment upon it. It is probably not too much to say that every mathematician is a poet at heart, and it is possible that every poet has more of the mathematical in his soul than he thinks. However this may be, it would not be without interest to study the tangible evidence of the relation of poetry to mathematics, and, indeed, of mathematics to all the various arts in which rhythm and symmetry and imagination enter. In such a study the productions of that interesting character, Professor Sylvester, would naturally be considered—not because they represent a high type of imaginative literature, but because they illustrate an interesting type of mind.

Among a large number of autograph letters of Professor Sylvester which I possess is one written on November 9, 1884, to W. J. C. Miller, Esq., mathematical editor of the *Educational Times*. This contains a poem entitled "Retrospect," which was published a little later. It shows the alterations which Professor Sylvester made and which, having never before appeared in print, may be welcomed by the few who have already seen the verses. The poem as a whole may be of interest to those who have attempted to solve what may be designated as Professor Sylvester's "other" problems. The verses and accompanying note are as follows: